

# ON THE CONFORMAL SYSTOLES OF FOUR-MANIFOLDS

M. J. D. HAMILTON

**ABSTRACT.** We extend a result of M. Katz on conformal systoles to all four-manifolds with  $b_2^+ = 1$  which have odd intersection form. The same result holds for all four-manifolds with  $b_2^+ = 1$  with even intersection form and which are symplectic or satisfy the so-called  $\frac{5}{4}$ -conjecture.

## 1. INTRODUCTION

There are several notions of systolic invariants for Riemannian manifolds, which were introduced by M. Berger and M. Gromov (see [14] and [3, 6] for an overview). The most basic concept is the  $k$ -systole  $\text{sys}_k(X, g)$  of a Riemannian manifold  $X$ , defined as the infimum over the volumes of all cycles representing non-zero classes in  $H_k(X; \mathbb{Z})$ . In this note we discuss a different systole, namely the *conformal systole*, which depends only on the conformal class of the Riemannian metric. We briefly review its definition (see Section 2 for details).

Let  $(X^{2n}, g)$  be a closed oriented even dimensional Riemannian manifold. The Riemannian metric defines an  $L^2$ -norm on the space of harmonic  $n$ -forms on  $X$  and hence induces a norm on the middle-dimensional cohomology  $H^n(X; \mathbb{R})$ . The *conformal  $n$ -systole*  $\text{confsys}_n(X, g)$  is the smallest norm of a non-zero element in the integer lattice  $H^n(X; \mathbb{Z})_{\mathbb{R}}$  in  $H^n(X; \mathbb{R})$ . It is known that for a fixed manifold  $X$  the conformal  $n$ -systoles are bounded from above as  $g$  varies over all Riemannian metrics. Hence the supremum  $CS(X) = \sup_g \text{confsys}_n(X, g)$  of the conformal systoles over all metrics  $g$  is a finite number, which is *a priori* a diffeomorphism invariant of  $X$ .

The interest in the literature has been to find bounds for  $CS(X)$  that depend only on the topology of  $X$ , e.g. the Euler characteristic of  $X$ , where  $X$  runs over some class of manifolds. In [5] P. Buser and P. Sarnak proved the following inequalities for the closed orientable surfaces  $\Sigma_s$  of genus  $s$ : there exists a constant  $C > 0$  independent of  $s$  such that

$$(1) \quad C^{-1} \log s < CS(\Sigma_s)^2 < C \log s, \quad \forall s \geq 2.$$

In dimension 4, M. Katz [15] proved a similar inequality for the conformal 2-systole of blow-ups of the complex projective plane  $\mathbb{C}P^2$ : there exists a constant  $C > 0$  independent of  $n$  such that

$$(2) \quad C^{-1} \sqrt{n} < CS(\mathbb{C}P^2 \# n \overline{\mathbb{C}P^2})^2 < Cn, \quad \forall n > 0.$$

In his proof, M. Katz used a conjecture on the *period map* of 4-manifolds  $X$  with  $b_2^+ = 1$ . The period map is defined as the map taking a Riemannian metric  $g$  to the point in the projectivization of the positive cone in  $H^2(X; \mathbb{R})$  given by the  $g$ -selfdual direction (see Section 2). The conjecture, which is still open, claims that this map is surjective. However, an inspection of the proof of M. Katz shows that this surjectivity conjecture in full strength is not needed and that in fact his theorem holds in much greater generality.

In Section 3, we first remark that the following proposition holds as a consequence of recent work of D. T. Gay and R. Kirby [12].

---

*Date:* November 16, 2006; MSC 2000: primary 53C23, 53D35, secondary 57M50, 57R57.

**Proposition 1.1.** *The period map for all closed 4-manifolds with  $b_2^+ = 1$  has dense image.*

Using the argument of M. Katz, this implies the following theorem.

**Theorem 1.1.** *There exists a universal constant  $C$  independent of  $X$  and  $n = b_2(X)$  such that*

$$(3) \quad C^{-1}\sqrt{n} < CS(X)^2 < Cn,$$

*for all closed 4-manifolds  $X$  with  $b_2^+ = 1$  which have odd intersection form.*

Another consequence of Proposition 1.1 is the following theorem.

**Theorem 1.2.** *Let  $X, X'$  be closed 4-manifolds with  $b_2^+ = 1$  which have isomorphic intersection forms. Then  $CS(X) = CS(X')$ .*

This shows that in dimension 4 the invariant  $CS$  is much coarser than a diffeomorphism invariant. Theorem 1.2 can be compared to a result of I. K. Babenko ([2], Theorem 8.1.), who showed that a certain 1-dimensional systolic invariant for manifolds of arbitrary dimension is a homotopy-invariant.

Theorem 1.2 enables us to deal with even intersection forms. Suppose  $X$  is a closed 4-manifold with  $b_2^+ = 1$  and even intersection form. By the classification of indefinite even quadratic forms, the intersection form of  $X$  is isomorphic to  $H \oplus (-k)E_8$  for some  $k \geq 0$ . In particular, for each  $r \in \mathbb{N}$  there are only finitely many possible even intersection forms of rank less or equal than  $r$ . Hence by Theorem 1.2, the invariant  $CS$  takes only finitely many values on all 4-manifolds with even intersection form,  $b_2^+ = 1$  and  $b_2 \leq r$ . We will show that symplectic 4-manifolds  $X$  with  $b_2^+(X) = 1$  and even intersection form necessarily have  $b_2(X) \leq 10$  (see Section 4). The same bound holds if  $X$  satisfies the so-called  $\frac{5}{4}$ -conjecture (see Section 5). Hence together with Theorem 1.1, we get the following corollary, which possibly covers all 4-manifolds with  $b_2^+ = 1$ .

**Corollary 1.1.** *There exists a universal constant  $C$  independent of  $X$  and  $n = b_2(X)$  such that*

$$(4) \quad C^{-1}\sqrt{n} < CS(X)^2 < Cn,$$

*for all closed 4-manifolds  $X$  with  $b_2^+ = 1$  which are symplectic or have odd intersection form  $Q$  or satisfy the  $\frac{5}{4}$ -conjecture if  $Q$  is even.*

## 2. DEFINITIONS

Let  $(X^{2n}, g)$  be a closed oriented Riemannian manifold. We denote the space of  $g$ -harmonic  $n$ -forms on  $X$  by  $\mathcal{H}^n(X)$ . The Riemannian metric defines an  $L^2$ -norm on  $\mathcal{H}^n(X)$  by

$$(5) \quad |\alpha|_{L^2}^2 = \int_X \alpha \wedge * \alpha, \quad \alpha \in \mathcal{H}^n(X),$$

where  $*$  is the Hodge operator.

Given the unique representation of cohomology classes by harmonic forms, we obtain an induced norm  $|\cdot|_g$ , which we call the  $g$ -norm, on the middle-dimensional cohomology  $H^n(X; \mathbb{R})$ . The conformal  $n$ -systole is defined by

$$(6) \quad \text{confsys}_n(X, g) = \min\{|\alpha|_g \mid \alpha \in H^n(X; \mathbb{Z})_{\mathbb{R}} \setminus \{0\}\},$$

where  $H^n(X; \mathbb{Z})_{\mathbb{R}}$  denotes the integer lattice in  $H^n(X; \mathbb{R})$ . More generally, if  $L$  is any lattice with a norm  $|\cdot|$ , we define

$$(7) \quad \lambda_1(L, |\cdot|) = \min\{|v| \mid v \in L \setminus \{0\}\},$$

hence  $\text{confsys}_n(X, g) = \lambda_1(H^n(X; \mathbb{Z})_{\mathbb{R}}, |\cdot|_g)$ . The conformal systole depends only on the conformal class of  $g$  since the Hodge star operator in the middle dimension is invariant under conformal changes of the metric.

The conformal systoles satisfy the following universal bound (see [15] equation (4.3)):

$$(8) \quad \text{confsys}_n(X, g)^2 < \frac{2}{3} b_n(X), \quad \text{for } b_n(X) \geq 2.$$

Clearly, there is also a bound for  $b_n(X) = 1$ , since the Hodge operator on harmonic forms is up to a sign the identity in this case, hence  $\text{confsys}_n(X, g) = 1$ . Therefore, the supremum

$$(9) \quad CS(X) = \sup_g \text{confsys}_n(X, g)$$

is well-defined for all closed orientable manifolds  $X^{2n}$ .

We now consider the case of 4-manifolds,  $n = 2$ . In this case the  $g$ -norm on  $H^2(X; \mathbb{R})$  is related to the intersection form  $Q$  by the following formula:

$$(10) \quad |\alpha|_g^2 = Q(\alpha^+, \alpha^+) - Q(\alpha^-, \alpha^-),$$

where  $\alpha = \alpha^+ + \alpha^-$  is the decomposition given by the splitting  $H^2(X; \mathbb{R}) = H^+ \oplus H^-$  into the subspaces represented by  $g$ -selfdual and anti-selfdual harmonic forms. We abbreviate this formula to

$$(11) \quad |\cdot|_g^2 = SR(Q, H^-),$$

where  $SR$  denotes sign-reversal. Since  $H^+$  is the  $Q$ -orthogonal complement of  $H^-$ , we conclude that the norm  $|\cdot|_g$  is completely determined by the intersection form and the  $g$ -anti-selfdual subspace  $H^-$ .

In particular, let  $X$  be a closed oriented 4-manifold with  $b_2^+ = \dim H^+ = 1$ . The map which takes a Riemannian metric to the selfdual line  $H^+$  in the cone  $\mathcal{P}$  of elements of positive square in  $H^2(X; \mathbb{R})$  (or to the point in the projectivization  $\mathbb{P}(\mathcal{P})$  of this cone) is called the period map. In the proof of his theorem, M. Katz used the following conjecture, which is still open, in the case of blow-ups of  $\mathbb{C}P^2$ .

**Conjecture 2.1.** *The period map is surjective for all closed oriented 4-manifolds with  $b_2^+ = 1$ .*

If  $X$  is a 4-manifold with  $b_2^+ = 1$ , we switch the orientation (this does not change the  $g$ -norm on  $H^2(X; \mathbb{R})$ ) to obtain  $b_2^- = 1$ . Then the  $g$ -norm is completely determined by the intersection form and the selfdual line, which in the new orientation is  $H^-$ .

**Lemma 2.1.** *Let  $\bar{X}$  be a 4-manifold with  $b_2^- = 1$  and intersection form  $\bar{Q}$  and let  $L$  be the integer lattice in  $H^2(\bar{X}; \mathbb{R})$ . Then  $\lambda_1(L, SR(\bar{Q}, V)^{1/2})$  depends continuously on the anti-selfdual line  $V$ .*

This follows because the vector space norm  $SR(\bar{Q}, V)^{1/2}$  depends continuously on  $V$  and the minimum in  $\lambda_1$  cannot jump (cf. Remark 9.1. in [15]).

### 3. PROOF OF PROPOSITION 1.1, THEOREM 1.1 AND THEOREM 1.2

The following theorem can be deduced from recent work of D. T. Gay and R. Kirby [12] (compare also [1]).

**Theorem 3.1.** *If  $X$  is a closed oriented 4-manifold and  $\alpha \in H^2(X; \mathbb{Z})_{\mathbb{R}}$  a class of positive square, then there exists a Riemannian metric on  $X$  such that the harmonic representative of  $\alpha$  is selfdual.*

This implies Proposition 1.1, because the set of points given by the lines through integral classes in  $H^2(X; \mathbb{R})$  form a dense subset of  $\mathbb{P}(\mathcal{P})$ . We can now prove Theorem 1.2.

*Proof.* Let  $\bar{X}$  be  $X$  with the opposite orientation,  $L$  be the integer lattice in  $H^2(\bar{X}; \mathbb{R})$  and  $\bar{Q} = -Q$ . We have

$$(12) \quad CS(X) \leq \sup_V \lambda_1(L, SR(\bar{Q}, V)^{1/2}),$$

where the supremum extends over all negative definite lines  $V$  in  $H^2(\bar{X}; \mathbb{R})$ . This inequality is an equality because the image of the period map is dense and because of Lemma 2.1. The right-hand side depends only on the intersection form.  $\square$

We now prove Theorem 1.1.

*Proof.* Let  $X$  be a closed 4-manifold with intersection form  $Q \cong (1) \oplus n(-1)$  for some  $n > 0$ . It is enough to prove inequalities of the form  $A\sqrt{n} < CS(X)^2 < Bn$  for some constants  $A, B > 0$ , since we can then take  $C = \max\{A^{-1}, B\}$ . The inequality on the right-hand side follows from equation (8). We are going to prove the inequality on the left, following the proof of M. Katz.

**Lemma 3.1.** *There exists a constant  $k(n) > 0$  (which depends only on  $n$  and is asymptotic to  $n/2\pi e$  for large  $n$ ) such that*

$$(13) \quad CS(X) \geq k(n)^{1/4}.$$

*Proof.* It is more convenient to work with  $\bar{X}$ , which is  $X$  with the opposite orientation. We identify  $H^2(\bar{X}; \mathbb{R}) = \mathbb{R}^{n,1} = \mathbb{R}^{n+1}$  with quadratic form  $q_{n,1}$  given by  $q_{n,1}(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2$ . If  $g$  is a metric on  $\bar{X}$  then the  $g$ -norm is given by  $|\cdot|_g^2 = SR(q_{n,1}, v)$  where  $*v = -v$  and  $SR$  means sign reversal in the direction of  $v$ .

Let  $L = I_{n,1} = \mathbb{Z}^{n,1} \subset \mathbb{R}^{n,1}$  be the integer lattice. According to Conway-Thompson (see [19], Ch. II, Theorem 9.5), there exists a positive definite odd integer lattice  $CT_n$  of rank  $n$  with

$$(14) \quad \min_{x \in CT_n \setminus \{0\}} x \cdot x \geq k(n),$$

where  $k(n)$  is asymptotic to  $n/2\pi e$  for  $n \rightarrow \infty$ . By the classification of odd indefinite unimodular forms,  $CT_n \oplus I_{0,1} \cong I_{n,1}$ , hence there exists a vector  $v \in \mathbb{Z}^{n,1}$  with  $q_{n,1}(v) = -1$  such that  $v^\perp \cong CT_n$ .

According to M. Katz, there exists an isometry  $A$  of  $(\mathbb{R}^{n,1}, q_{n,1})$  such that

$$(15) \quad \lambda_1(L, SR(q_{n,1}, Av)^{1/2}) \geq k(n)^{1/4}.$$

By Proposition 1.1, there exists a sequence of Riemannian metrics  $g_i$  on  $X$  whose selfdual lines converge to the line through  $Av$ . Lemma 2.1 implies

$$(16) \quad \text{confsys}_n(X, g_i) \xrightarrow{i \rightarrow \infty} \lambda_1(L, SR(q_{n,1}, Av)^{1/2}).$$

Hence  $CS(X) \geq k(n)^{1/4}$ .  $\square$

Lemma 3.1 finishes the proof of Theorem 1.1.  $\square$

## 4. SYMPLECTIC MANIFOLDS

We now show that symplectic 4-manifolds with  $b_2^+ = 1$  necessarily have  $b_2 \leq 10$ , as stated in the introduction (note that we always assume symplectic forms to be compatible with the orientation, i.e. of positive square).

**Lemma 4.1.** *Let  $X$  be a closed symplectic 4-manifold with  $b_2^+ = 1$  and even intersection form  $Q$ . Then  $Q \cong H$  or  $Q \cong H \oplus (-E_8)$ .*

Here  $H$  denotes the bilinear form given by  $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $E_8$  is a positive-definite, even form of rank 8 associated to the Dynkin diagram of the Lie group  $E_8$  (see [13]).

*Proof.* If  $b_2^- = 0$  then  $Q = (1)$ . If  $b_2^- > 0$  then  $Q$  is indefinite and hence of the form  $Q = H \oplus (-k)E_8$ , since  $Q$  is even. It follows that  $X$  is minimal because the intersection form does not split off a  $(-1)$ . If  $K^2 < 0$  then according to a theorem of A.-K. Liu [18],  $X$  is an irrational ruled surface and hence has intersection form  $Q = H$  (or  $Q = (1) \oplus (-1)$ , which is odd). If  $K^2 = 2\chi + 3\sigma \geq 0$  then  $4b_1 + b_2^- \leq 9$  and  $b_1 = 0$  or  $b_1 = 2$ , because  $1 - b_1(X) + b_2^+(X)$  is an even number for every almost complex 4-manifold  $X$ . If  $b_1 = 0$  then  $b_2^- \leq 9$ , hence  $Q = H$  or  $H \oplus (-E_8)$ . If  $b_1 = 2$  then  $b_2^- \leq 1$ , hence  $Q = H$ .  $\square$

*Remark 4.1.* It is possible to give a different proof of Proposition 1.1 for symplectic manifolds, which relies on a theorem of T.-J. Li and A.-K. Liu ([17], Theorem 4). This theorem implies that on a closed 4-manifold  $X$  with  $b_2^+ = 1$  which admits a symplectic structure, the set of classes in  $H^2(X; \mathbb{R})$  represented by symplectic forms is dense in the positive cone, because it is the complement of at most countably many hyperplanes. If a closed symplectic 4-manifold with  $b_2^+ = 1$  is *minimal* (i.e. there are no symplectic  $(-1)$ -spheres), then the period map is in fact surjective.

5. THE  $\frac{5}{4}$ -CONJECTURE AND SOME EXAMPLES

The  $\frac{5}{4}$ -conjecture is a weak analogue of the  $\frac{11}{8}$ -conjecture which relates the signature and second Betti number of spin 4-manifolds. The main result in this direction is a theorem of M. Furuta [11], generalizing work of S. K. Donaldson [7, 8], that all closed oriented spin 4-manifolds  $X$  with  $b_2(X) > 0$  satisfy the inequality

$$(17) \quad \frac{5}{4}|\sigma(X)| + 2 \leq b_2(X),$$

where  $\sigma(X)$  denotes the signature. C. Bohr [4] then proved a slightly weaker inequality  $\frac{5}{4}|\sigma(X)| \leq b_2(X)$  for all 4-manifolds with even intersection form and certain fundamental groups, including all finite and all abelian groups. These are special instances of the following general  $\frac{5}{4}$ -conjecture.

**Conjecture 5.1.** *If  $X$  is a closed oriented even 4-manifold, then*

$$(18) \quad \frac{5}{4}|\sigma(X)| \leq b_2(X),$$

where  $\sigma(X)$  denotes the signature.

Here we call a 4-manifold *even*, if it has even intersection form.

*Remark 5.1.* J.-H. Kim [16] has proposed a proof of the  $\frac{5}{4}$ -conjecture. However, some doubts have been raised about the validity of the proof. Therefore, we have chosen to state the result still as a conjecture.

**Lemma 5.1.** *If  $X$  is an even 4-manifold with  $b_2^+ = 1$ , then the  $\frac{5}{4}$ -conjecture holds for  $X$  if and only if  $Q \cong H$  or  $Q \cong H \oplus (-E_8)$ .*

*Proof.* If  $X$  is an even 4-manifold with  $b_2^+ = 1$ , then  $Q \cong H \oplus (-k)E_8$  for some  $k \geq 0$ . The  $\frac{5}{4}$ -conjecture is equivalent to  $k \leq 1$ .  $\square$

In particular, by Lemma 4.1, the  $\frac{5}{4}$ -conjecture holds for all even symplectic 4-manifolds with  $b_2^+ = 1$ .

There are many examples of 4-manifolds with  $b_2^+ = 1$  where Theorem 1.2 applies, e.g. the infinite family of simply-connected pairwise non-diffeomorphic Dolgachev surfaces which are all homeomorphic to  $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$  (see [13]). These 4-manifolds are Kähler, hence symplectic. There are also recent constructions of infinite families of non-symplectic and pairwise non-diffeomorphic 4-manifolds homeomorphic to  $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$  for  $n \geq 5$  (see [10, 20]). If we take multiple blow-ups of these manifolds, the blow-up formula for the Seiberg-Witten invariants [9] shows that the resulting manifolds stay pairwise non-diffeomorphic. Hence we obtain infinite families of symplectic and non-symplectic 4-manifolds  $X$  with  $n = b_2(X) \rightarrow \infty$ , where Theorem 1.1 applies.

**Acknowledgments.** I wish to thank D. Kotschick for raising the question about possible extensions of the work of M. Katz and many helpful comments.

## REFERENCES

- [1] D. Auroux, S. K. Donaldson and L. Katzarkov, *Singular Lefschetz pencils*, Geometry and Topology **9**, 1043–1114 (2005).
- [2] I. K. Babenko, *Asymptotic invariants of smooth manifolds*, Izv. Ross. Akad. Nauk. Ser. Mat. **56**, 707–751 (1992) (Russian); Engl. transl. in Russian Acad. Sci. Izv. Math. **41**, 1–38 (1993).
- [3] M. Berger, *Systoles et applications selon Gromov*, Séminaire Bourbaki, 45ème année, 1992-93, no. 771, Astérisque **216**, 195–220 (1993).
- [4] C. Bohr, *On the signatures of even 4-manifolds*, Math. Proc. Cambridge Philos. Soc. **132**, 453–469 (2002).
- [5] P. Buser and P. Sarnak, *On the period matrix of a Riemann surface of large genus, with an appendix by J. H. Conway and N. J. A. Sloane*, Invent. Math. **117**, 27–56 (1994).
- [6] C. B. Croke and M. Katz, *Universal volume bounds in Riemannian manifolds*, Surveys in Differential Geometry VIII, ed. S.-T. Yau. Somerville, MA. International Press 2003.
- [7] S. K. Donaldson, *Connections, cohomology and the intersection forms of 4-manifolds*, J. Differential Geom. **24**, 275–341 (1986).
- [8] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26**, 397–428 (1987).
- [9] R. Fintushel and R. J. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19**, 145–157 (1995).
- [10] R. Fintushel and R. J. Stern, *Double node neighborhoods and families of simply connected 4-manifolds with  $b^+ = 1$* , J. Amer. Math. Soc. **19**, 171–180 (2006).
- [11] M. Furuta, *Monopole equation and the  $\frac{11}{8}$ -conjecture*, Math. Res. Lett. **8**, 279–291 (2001).
- [12] D. T. Gay and R. Kirby, *Constructing symplectic forms on 4-manifolds which vanish on circles*, Geometry and Topology **8**, 743–777 (2004).
- [13] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. Providence, Rhode Island. American Mathematical Society 1999.
- [14] M. Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18**, 1–147 (1983).
- [15] M. Katz, *Four-manifold systoles and surjectivity of period map*, Comment. Math. Helv. **78**, 772–768 (2003).
- [16] J.-H. Kim, *The  $\frac{10}{8}$ -conjecture and equivariant  $e_C$ -invariants*, Math. Ann. **329**, 31–47 (2004).
- [17] T.-J. Li and A.-K. Liu, *Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with  $b^+ = 1$* , J. Differential Geom. **58**, 331–370 (2001).

- [18] A.-K. Liu, *Some new applications of general wall crossing formula, Gompf's conjecture and its applications*, Math. Res. Lett. **3**, 569–585 (1996).
- [19] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, New York-Heidelberg, Springer 1973.
- [20] J. Park, A. I. Stipsicz and Z. Szabó, *Exotic smooth structures on  $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$* , Math. Res. Lett. **12**, 701–712 (2005).

MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, 80333 MÜNCHEN, GERMANY

*E-mail address:* Mark.Hamilton@mathematik.uni-muenchen.de